Double Field Theory as the Stringy Extension of Einstein Gravity

Jeong-Hyuck Park 朴 廷 爀 박 정 혁

Sogang University, Seoul

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Prologue

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- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
  - Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  - $\nabla_\lambda g_{\mu\nu} = 0,\ \Gamma^\lambda_{[\mu\nu]} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  - Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_{\kappa\lambda\mu\nu} \longrightarrow R$
- On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal footing, as they, *i.e.* NS-NS sector fields form a 'multiplet of T-duality'.
- This suggests the existence of a novel unifying geometric description of them, generalizing the above Riemannian formalism.
- Basically, Riemannian geometry is for *Particle* theory. *String* theory requires a novel differential geometry which geometrizes the whole NS-NS sector.

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$$\int \mathrm{d}^D x \, \sqrt{-|g|} \, e^{-2\phi} \left(R + 4 \, |\mathrm{d}\phi|^2 - \tfrac{1}{12} \, |\mathrm{d}B|^2 \right),$$

where the last two terms correspond to the kinetic terms of the dilaton and the B-field.

- In this conventional description, the Riemannian metric provides the background geometry, while the dilaton and the B-field are viewed as *'matter'* living on it.

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- This talk introduces a stringy geometry for the massless NS-NS sector under the name,

'Semi-covariant Geometry'

which underlies

Double Field Theory (DFT) = Stringy Extension of Einstein's Gravity .

Siegel; Hull, Zwiebach, Hohm

- Contrary to what it may sound like, the semi-covariant geometry is a **completely covariant** approach to DFT, as it manifests simultaneously for every term in formulas:

  - **O**(D, D) **T-duality**
  - **DFT-diffeomorphisms (generalized Lie derivative)**
  - **A pair of local Lorentz symmetries, Spin**$(1, D-1)_L \times$ **Spin**$(D-1, 1)_R$
  - **Other gauge symmetries,** *e.g.* **Yang-Mills or Higher Spin**

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- **This talk reviews a series of my works in collaborations with**

  Imtak Jeon, Kanghoon Lee, Yoonji Suh,

  Chris Blair, Emanuel Malek, Wonyoung Cho, Jose Fernández-Melgarejo,

  Soo-Jong Rey, Woohyun Rim, Yuho Sakatani,

  Sung Moon Ko, Charles Melby-Thompson, Rene Meyér,

  Kang-Sin Choi (Phenomenologist) and Xavier Bekaert:

- **1011.1324, 1102.0419,**
  1105.6294, 1109.2035, 1112.0069,
  1206.3478, 1210.5078,
  1302.1652, 1304.5946, 1307.8377, 1311.5109,
  1402.5027,
  1505.01301, 1506.05277, 1507.07545, 1508.01121,
  1605.00403.

Talk contents

- **Doubled-yet-gauged coordinates**
- **Worldsheet perspective**
- **Target spacetime perspective**
- **Phenomenological implications, especially to the Standard Model**
- **Higher Spin Double Field Theory**

- ***Notation***

Capital letters denote the $\mathbf{O}(D,D)$ vector indices, *i.e.* $A,B,C,\cdots=1,2,\cdots,D{+}D$, which can be freely raised or lowered by the $\mathbf{O}(D,D)$ invariant constant metric,

$$\mathcal{J}_{AB}=\left(\begin{array}{cc}0&1\\1&0\end{array}\right)$$

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$$\mathcal{J}_{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

With $D \equiv 10$ for SUSY,

Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	$\mathbf{O}(10,10)$ & DFT-diffeom. vector	$\mathcal{J}_{AB}$
$p, q, \cdots$	$\mathbf{Spin}(1,9)_L$ vector	$\eta_{pq} = \mathrm{diag}(-++\cdots+)$
$\alpha, \beta, \cdots$	$\mathbf{Spin}(1,9)_L$ spinor	$C_{+\alpha\beta}, \qquad (\gamma^p)^T = C_+\gamma^p C_+^{-1}$
$\bar{p}, \bar{q}, \cdots$	$\mathbf{Spin}(9,1)_R$ vector	$\bar{\eta}_{\bar{p}\bar{q}} = \mathrm{diag}(+--\cdots-)$
$\bar{\alpha}, \bar{\beta}, \cdots$	$\mathbf{Spin}(9,1)_R$ spinor	$\bar{C}_{+\bar{\alpha}\bar{\beta}}, \qquad (\bar{\gamma}^{\bar{p}})^T = \bar{C}_+\bar{\gamma}^{\bar{p}}\bar{C}_+^{-1}$

- ***Doubled-yet-gauged coordinates***

The spacetime coordinates are formally doubled, being $(D+D)$-dimensional. However, **the doubled coordinates need to be gauged**: the coordinate space is equipped with an equivalence relation,

$$x^{A} \sim x^{A} + \phi\partial^{A}\varphi\,,$$

which we call **'Coordinate Gauge Symmetry'**.

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Each equivalence class, or gauge orbit, represents a single physical point.

Diffeomorphism symmetry means an invariance under arbitrary reparametrizations of the gauge orbits.

$*$ *The claim is that D-dimensional spacetime can be better understood in terms of the doubled-yet-gauged $(D{+}D)$ number of coordinates, at least for String Theory.*

- *Realization of the coordinate gauge symmetry.*

The equivalence relation is realized in DFT by enforcing that, arbitrary functions and their arbitrary derivatives, denoted here collectively by Φ, are invariant under the coordinate gauge symmetry shift,

$$\Phi(x+\Delta)=\Phi(x)\,, \qquad \Delta^{A}=\phi\partial^{A}\varphi\,.$$

- *Section condition.*

The invariance under the coordinate gauge symmetry can be easily shown to be equivalent to the section condition:

$$\textbf{Coordinate Gauge Symmetry} \quad \Longleftrightarrow \quad \partial_{A}\partial^{A}\equiv 0\,.$$

JHP, Lee-JHP 2013

Explicitly, the section condition implies

$$\begin{array}{ll}\partial^{A}\varphi\partial_{A}\Phi=0 & \quad (\text{strong constraint})\,,\\ \partial_{A}\partial^{A}\Phi=0 & \quad (\text{weak constraint})\,.\end{array}$$

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- *Diffeomorphisms.*

Diffeomorphism symmetry in $\mathbf{O}(D,D)$ DFT is generated by a generalized Lie derivative Siegel, Courant, Grana

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B\partial_B T_{A_1\cdots A_n} + \omega_T\,\partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}X_B - \partial_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}\,,$$

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where ω_T denotes the weight.

In particular, the generalized Lie derivative of the $\mathbf{O}(D,D)$ invariant metric is trivial,

$$\hat{\mathcal{L}}_X \mathcal{J}_{AB} = 0\,.$$

The commutator is closed by C-bracket Hull-Zwiebach

$$\left[\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y\right] = \hat{\mathcal{L}}_{[X,Y]_{\rm C}}\,, \qquad [X,Y]^A_{\rm C} = X^B\partial_B Y^A - Y^B\partial_B X^A + \tfrac{1}{2}Y^B\partial^A X_B - \tfrac{1}{2}X^B\partial^A Y_B\,.$$

- *Diffeomorphisms.*

  - Hohm-Zwiebach ansatz for finite transformations:

  $$F := \tfrac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, \qquad\qquad \bar{F} := \mathcal{J}F^{t}\mathcal{J}^{-1} = F^{-1}\,,$$

  where

  $$L_{M}{}^{N} := \partial_{M}x'^{N}\,, \qquad\qquad \bar{L} := \mathcal{J}L^{t}\mathcal{J}^{-1}\,.$$

  - Though nice and compact, F does not precisely coincide with "$\mathbf{exp}(\hat{\mathcal{L}}_{X})$".

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Worldsheet perspective: string action

1304.5946/1307.8377

Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- In the doubled-yet-gauged coordinate system, the usual infinitesimal one-form, $\mathrm{d}x^M$, is **not** a covariant vector of DFT: it does not transform covariantly under DFT-diffeomorphisms (obeying the way the 'generalized Lie derivative' would dictate).

- Hence, $\mathrm{d}x^M \mathrm{d}x^N \mathcal{H}_{MN}$ can **not** give a 'proper length' in DFT.

- Further, it is **not** coordinate gauge symmetry invariant,

$$\mathrm{d}x^M \quad \longrightarrow \quad \mathrm{d}(x^M + \phi\partial^M\varphi) \;\neq\; \mathrm{d}x^M\,.$$

- **These can be all cured by introducing a gauged infinitesimal one-form,**

$$Dx^M := \mathrm{d}x^M - \mathcal{A}^M\,.$$

  The gauge potential should satisfy the same property as the coordinate gauge symmetry generator, such that

$$\mathcal{A}^M\partial_M = 0\,, \qquad\qquad \mathcal{A}_M\mathcal{A}^M = 0\,,$$

  or suggestively the 'gauged' section condition,

$$(\partial_M + \mathcal{A}_M)(\partial^M + \mathcal{A}^M) = 0\,.$$

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Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- Under coordinate gauge symmetry, we have the invariance of Dx^M,

$$\begin{array}{lll} x^M & \longrightarrow & x'^M = x^M + \phi\partial^M\varphi\,, \\ \mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^M + \mathrm{d}(\phi\partial^M\varphi) \qquad : \quad \mathcal{A}'^M\partial'_M = 0\,, \\ Dx^M & \longrightarrow & D'x'^M = Dx^M = \mathrm{d}x^M - \mathcal{A}^M\,. \end{array}$$

- Similarly, under (finite) DFT diffeomorphisms *à la* Hohm-Zwiebach

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we have the covariance,

$$\begin{array}{lll} x^M & \longrightarrow & x'^M(x)\,, \\ \mathcal{H}_{MN}(x) & \longrightarrow & \mathcal{H}'_{MN}(x') = \bar{F}_M{}^K\bar{F}_N{}^L\mathcal{H}_{KL}(x)\,, \\ \mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N(L-F)_N{}^M \qquad : \quad \mathcal{A}'^M\partial'_M = 0\,, \\ Dx^M & \longrightarrow & D'x'^M = Dx^N F_N{}^M\,. \end{array}$$

Doubled-yet-gauged coordinates & Gauged infinitesimal one-form

- Under coordinate gauge symmetry, we have the invariance of Dx^M,

$$\begin{array}{ccll} x^M & \longrightarrow & x'^M = x^M + \phi\partial^M\varphi\,, & \\ \mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^M + \mathrm{d}(\phi\partial^M\varphi) & : \quad \mathcal{A}'^M\partial'_M = 0\,, \\ Dx^M & \longrightarrow & D'x'^M = Dx^M = \mathrm{d}x^M - \mathcal{A}^M\,. & \end{array}$$

- Similarly, under (finite) DFT diffeomorphisms *à la* Hohm-Zwiebach

$$\begin{array}{ll} L_M{}^N := \partial_M x'^N\,, & \bar{L} := \mathcal{J}L^t\mathcal{J}^{-1}\,, \\ F := \frac{1}{2}\left(L\bar{L}^{-1} + \bar{L}^{-1}L\right)\,, & \bar{F} := \mathcal{J}F^t\mathcal{J}^{-1} = F^{-1}\,, \end{array}$$

we have the covariance,

$$\begin{array}{rcll} x^M & \longrightarrow & x'^M(x)\,, & \\ \mathcal{H}_{MN}(x) & \longrightarrow & \mathcal{H}'_{MN}(x') = \bar{F}_M{}^K\bar{F}_N{}^L\mathcal{H}_{KL}(x)\,, & \\ \mathcal{A}^M & \longrightarrow & \mathcal{A}'^M = \mathcal{A}^N F_N{}^M + \mathrm{d}X^N(L-F)_N{}^M & : \quad \mathcal{A}'^M\partial'_M = 0\,, \\ Dx^M & \longrightarrow & D'x'^M = Dx^N F_N{}^M\,. & \end{array}$$

Fixing the coordinate gauge symmetry

- In DFT (unlike EFT), the solution of the section condition, *i.e.* the section is unique up to the duality rotations,

$$\frac{\partial}{\partial x^M} = \left(\frac{\partial}{\partial \tilde{x}_\mu}\,,\,\frac{\partial}{\partial x^\nu}\right) \equiv \left(0\,,\,\frac{\partial}{\partial x^\nu}\right) \qquad : \text{ Conventional choice of the section}$$

- Then, the 'coordinate gauge symmetry' reads

$$(\,\tilde{x}_\mu\;,\;x^\nu\,)\;\sim\;(\,\tilde{x}_\mu + \phi\partial_\mu\varphi\;,\;x^\nu\,)\,.$$

- The coordinate gauge potential and the gauged infinitesimal one-form become

$$\mathcal{A}^M = A_\lambda\partial^M x^\lambda = (\,A_\mu\;,\;0\,)\,, \qquad Dx^M = (\,\mathrm{d}\tilde{x}_\mu - A_\mu\;,\;\mathrm{d}x^\nu\,)\,.$$

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Side remark: Newton mechanics with doubled-yet-gauged coordinates

- The doubled-yet-gauged coordinates can be applied to any physical system, not exclusively to DFT.

- Newton mechanics can be formulated on the doubled-yet-gauged space, $x^M = (\tilde{x}_m, x^n)$,

$$\mathcal{L}_{\rm Newton} = \frac{1}{2} m \, D_t x^M D_t x^N \, \delta_{MN} - V(x) \,,$$

where $M, N = 1, 2, \cdots, 6$ and the potential, $V(x)$, satisfies the section condition.

- With the conventional choice of the section, we get

$$\mathcal{L}_{\rm Newton} = \frac{1}{2} m \, \dot{x}^m \dot{x}^n \, \delta_{mn} - V(x) + \frac{1}{2} m \left(\dot{\tilde{x}}_m - A_m \right) \left(\dot{\tilde{x}}_n - A_n \right) \delta^{mn} \,.$$

Hence, after integrating out A_m, we recover the conventional formulation.

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String probes the doubled-yet-gauged spacetime

- **DFT string action** is with $D_iX^M = \partial_i X^M - \mathcal{A}_i^M$, JHP-Lee 2013, *c.f.* Hull 2006

$$\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}_{\rm string}\,, \qquad \mathcal{L}_{\rm string} = -\tfrac{1}{2}\sqrt{-h}\,h^{ij}D_iX^M D_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

- The action is **fully symmetric** for an arbitrary curved generalized metric, $\mathcal{H}_{MN}(X)$, essentially due to the auxiliary coordinate gauge potential, $\mathcal{A}_i^M$, under
  - worldsheet diffeomorphisms plus Weyl symmetry
  - $\mathbf{O}(D,D)$ T-duality
  - target spacetime DFT-diffeomorphisms
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DFT backgrounds : Riemannian vs. non-Riemannian

- W.r.t. the conventional choice of the section, $\frac{\partial}{\partial\tilde{x}_\mu}\equiv 0$, **'Riemannian' generalized metric** assumes the well-known form,

$$\mathcal{H}_{AB}=\left(\begin{array}{cc} g^{-1} & -g^{-1}B \\ Bg^{-1} & g-Bg^{-1}B \end{array}\right).$$

Up to field redefinition (*e.g.* β-gravity Andriot-Betz) this is the most general form of a symmetric $\mathbf{O}(D,D)$ element, **if the upper left $D\times D$ block is 'non-degenerate'.**

- **The DFT sigma model then reduces to the standard string action,**

$$\tfrac{1}{4\pi\alpha'}\mathcal{L}_{\rm string}\equiv\tfrac{1}{2\pi\alpha'}\left[-\tfrac{1}{2}\sqrt{-h}h^{ij}\partial_iX^\mu\partial_jX^\nu g_{\mu\nu}(X)+\tfrac{1}{2}\epsilon^{ij}\partial_iX^\mu\partial_jX^\nu B_{\mu\nu}(X)+\tfrac{1}{2}\epsilon^{ij}\partial_i\tilde{X}_\mu\partial_jX^\mu\right],$$

with the bonus of the topological term introduced by Giveon-Rocek; Hull.

- The EOM of $\mathcal{A}_i^M$ implies **self-duality** on the full doubled spacetime,

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- On the other hand, w.r.t. $\frac{\partial}{\partial\tilde{x}_\mu} \equiv 0$ again, the **'non-Riemannian' DFT background** is characterized by the 'degenerate' upper left $D \times D$ block, such that it does not admit any Riemannian interpretation, even locally.

- It turns out that **upon such a non-Riemannian DFT background, the DFT sigma model reduces to a 'chiral' action for the 'untilde' coordinates, X^μ.**

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Non-Riemannian DFT background for non-relativistic string theory

- Less simple example of a **non-Riemannian DFT background**, which is supersymmetric, can be obtained by performing a doubly T-dual rotation, $(t, x^1) \Leftrightarrow (\tilde{t}, \tilde{x}_1)$, of the known F1 background (Dabholkar-Gibbons-Harvey-Ruiz 1990): With $D = 10 = 2 + 8$,

$$\mathcal{H}_{MN} = \begin{pmatrix} 0 & 0 & \epsilon^{\alpha}{}_{\beta} & 0 \\ 0 & \delta^{ij} & 0 & 0 \\ -\epsilon_{\alpha}{}^{\beta} & 0 & f\eta_{\alpha\beta} & 0 \\ 0 & 0 & 0 & \delta_{ij} \end{pmatrix}, \qquad f = 1 + \frac{Q}{r^6}\,, \qquad r^2 = \textstyle\sum_{i=2}^{9}(x^i)^2\,.$$

- Upon this non-Riemannian background with $Q \equiv 0$, the DFT sigma model reduces precisely to **the non-relativistic string theory action** by Gomis-Ooguri 2000.

  Further, the non-relativistic sigma model **spectrum matches with the 'perturbation'** of DFT. This is one clear example which shows that DFT is not a mere reformulation but a nontrivial extension of SUGRA. Ko-Melby-Thompson-Meyer-JHP 2015

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- The Euler-Lagrange equation for $X^{M}(\sigma)$ gives the **Stringy Geodesic Motion:**

$$\frac{1}{\sqrt{-h}}\partial_i\left(\sqrt{-h}D^iX^M\mathcal{H}_{ML}\right)-\Gamma_{LMN}(PD_iX)^M(\bar{P}D^iX)^N=0$$

where we set

$$(PD_iX)^M=P^M{}_ND_iX^N\,,\qquad P_{MN}=\tfrac{1}{2}(\mathcal{J}_{MN}+\mathcal{H}_{MN})\,,$$
$$(\bar{P}D_iX)^M=\bar{P}^M{}_ND_iX^N\,,\qquad \bar{P}_{MN}=\tfrac{1}{2}(\mathcal{J}_{MN}-\mathcal{H}_{MN})\,,$$

and Γ_{LMN} **corresponds to the DFT extension of the Christoffel connection.**

For the exposition, we turn to the target spacetime perspective henceforth.

Target spacetime perspective: (S)DFT

1011.1324/1105.6294/1210.5078/1505.01301/1506.05277

- *Dilaton and a pair of two-index projectors.*

The **geometric objects** in DFT consist of a **dilaton, d,** and a pair of symmetric **projection operators,**

$$P_{AB} = P_{BA}\,, \qquad \bar{P}_{AB} = \bar{P}_{BA}\,, \qquad P_A{}^B P_B{}^C = P_A{}^C\,, \qquad \bar{P}_A{}^B \bar{P}_B{}^C = \bar{P}_A{}^C\,.$$

Further, the projectors are orthogonal and complementary,

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Remark: The difference of the two projectors, $P_{AB}-\bar{P}_{AB}=\mathcal{H}_{AB}$, corresponds to the "generalized metric". Yet, in supersymmetric double field theories, it appears that the projectors are more fundamental than the "generalized metric", as there are a pair of vielbeins which are "square-roots" of the projectors .

- *Integral measure.*

  - While the projectors are weightless, the dilaton gives rise to the $\mathbf{O}(D,D)$ invariant integral measure with weight one, after exponentiation,

$$e^{-2d}\,.$$

  - Naturally **the cosmological constant term in DFT** is given by

$$e^{-2d}\Lambda_{\rm DFT}$$

    which deviates from the conventional one in Riemannian GR, and hence reformulates the **cosmological constant problem** in a novel manner.

    Jeon-Lee-JHP 2011

    c.f. Meissner-Veneziano 1991

  - *Remark: Scherk-Schwarz-type dimensional reductions from $D=10$ half-maximal SDFT can produce $\Lambda_{\rm DFT}>0$ (as well as $\Lambda_{\rm DFT}<0$),*

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- *Semi-covariant derivative and semi-covariant four-index curvature.*

We define a semi-covariant derivative,

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega_T \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,,$$

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and a semi-covariant four-index curvature,

$$S_{ABCD} := \tfrac{1}{2} \left(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB} \Gamma_{ECD} \right) .$$

Here R_{ABCD} denotes the ordinary "field strength" of a connection,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED} \quad \Leftarrow \ \mathrm{d}\Gamma + \Gamma \wedge \Gamma .$$

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As will be explained below, one can determine the (torsionelss) connection uniquely:

$$\begin{aligned}\Gamma_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) ,\end{aligned}$$

which corresponds to **the DFT generalization of the Christoffel connection, 1105.6294** .

A crucial **defining property of the semi-covariant four-index curvature** is that, under arbitrary transformation of the connection, it transforms as total derivative,

$$\delta S_{ABCD} = \nabla_{[A}\delta\Gamma_{B]CD} + \nabla_{[C}\delta\Gamma_{D]AB}\,.$$

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Further, the semi-covariant four-index curvature satisfies precisely the same symmetric properties as the ordinary Riemann curvature,

$$S_{ABCD} = S_{[AB][CD]} = S_{CDAB}\,, \qquad\qquad S_{[ABC]D} = 0\,,$$

as well as additional identities involving the projectors,

$$P_I{}^A P_J{}^B \bar{P}_K{}^C \bar{P}_L{}^D S_{ABCD}{=}0\,, \qquad\qquad P_I{}^A \bar{P}_J{}^B P_K{}^C \bar{P}_L{}^D S_{ABCD} = 0\,.$$

Notably, it turns out that the naive scalar contraction vanishes identically,

$$S^{AB}{}_{AB} = 0\,.$$

- *The uniqueness of the DFT Christoffel connection.*

The connection is the unique solution to the following four constraints:

$$\nabla_A P_{BC} = 0\,, \qquad \nabla_A \bar{P}_{BC} = 0\,,$$
$$\nabla_A d = -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^{B}{}_{BA} = 0\,,$$
$$\Gamma_{ABC} + \Gamma_{BCA} + \Gamma_{CAB} = 0\,,$$
$$\mathcal{P}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF}\Gamma_{DEF} = 0\,.$$

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- The first two relations are the compatibility conditions with all the geometric objects in DFT , or the massless NS-NS sector, and further imply $\nabla_A \mathcal{J}_{BC} = 0$.

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  - The next cyclic property makes the semi-covariant derivative compatible with the generalized Lie derivative, as well as with the C-bracket,

$$\hat{\mathcal{L}}_X(\partial) = \hat{\mathcal{L}}_X(\nabla)\,, \qquad [X,Y]_{\mathrm{C}}(\partial) = [X,Y]_{\mathrm{C}}(\nabla)\,.$$

  The relaxation of the cyclic condition leads to a torsionful connection which can be used to ensure the 1.5 formalism in the full order supersymmetric completions of DFT. 1112.0069 (Half-maximal SUSY) / 1210.5078 (Maximal SUSY)

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- The last formulae are 'projection conditions' which ensure the uniqueness.

- Explicitly the **six-index projection operators** are

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}\,, \qquad \mathcal{P}_{ABC}{}^{DEF}\mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{ABC}{}^{GHI}\,,$$
$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D}\,, \qquad \bar{\mathcal{P}}_{ABC}{}^{DEF}\bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{ABC}{}^{GHI}\,.$$

They are symmetric and traceless,

$$\mathcal{P}_{ABCDEF} = \mathcal{P}_{DEFABC}\,, \qquad \mathcal{P}_{ABCDEF} = \mathcal{P}_{A[BC]D[EF]}\,, \qquad P^{AB}\mathcal{P}_{ABCDEF} = 0\,,$$
$$\bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{DEFABC}\,, \qquad \bar{\mathcal{P}}_{ABCDEF} = \bar{\mathcal{P}}_{A[BC]D[EF]}\,, \qquad \bar{P}^{AB}\bar{\mathcal{P}}_{ABCDEF} = 0\,.$$

Remark: Failure of the Equivalence Principle

Unlike the usual Christoffel symbol in Riemannian geometry, the DFT-diffeomorphisms cannot transform our connection to vanish point-wise:

$$\begin{aligned}\Gamma_{CAB} \quad &= 2\left(P\partial_{C}P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E} - P_{[A}{}^{D}P_{B]}{}^{E}\right)\partial_{D}P_{EC} \\ &\quad - \tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^{D} + P_{C[A}P_{B]}{}^{D}\right)\left(\partial_{D}d + (P\partial^{E}P\bar{P})_{[ED]}\right) \\ &\neq 0\,.\end{aligned}$$

That is to say, there is no 'normal' coordinate in DFT.

This can be viewed as the failure of the equivalence principle applied to an extended object, *i.e.* string.

- Crucially, **the projection operator dictates the anomalous terms** in the diffeomorphic transformations of the semi-covariant derivative and the four-index curvature,

$$(\delta_X-\hat{\mathcal{L}}_X)\nabla_C T_{A_1\cdots A_n}=\sum_{i=1}^{n}2(\mathcal{P}+\bar{\mathcal{P}})_{CA_i}{}^{BDEF}\partial_D\partial_E X_F\ T_{A_1\cdots A_{i-1}BA_{i+1}\cdots A_n}\,,$$

$$(\delta_X-\hat{\mathcal{L}}_X)S_{ABCD}=2\nabla_{[A}\Big((\mathcal{P}+\bar{\mathcal{P}})_{B][CD]}{}^{EFG}\partial_E\partial_F X_G\Big)+2\nabla_{[C}\Big((\mathcal{P}+\bar{\mathcal{P}})_{D][AB]}{}^{EFG}\partial_E\partial_F X_G\Big).$$

- *Complete covariantizations.*

Both the semi-covariant derivative and the semi-covariant four-index curvature can be fully covariantized, through appropriate contractions with the projectors:

$$
\begin{array}{lll}
P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1\cdots B_n}\,, & \bar{P}_{C}{}^{D}P_{A_1}{}^{B_1}\cdots P_{A_n}{}^{B_n}\nabla_{D}T_{B_1\cdots B_n}\,, & \\
P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1\cdots D_n}\,, & \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1\cdots D_n} & \text{(divergences)}\,,\\
P^{AB}\bar{P}_{C_1}{}^{D_1}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1\cdots D_n}\,, & \bar{P}^{AB}P_{C_1}{}^{D_1}\cdots P_{C_n}{}^{D_n}\nabla_{A}\nabla_{B}T_{D_1\cdots D_n} & \text{(Laplacians)}\,,
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$$
\begin{array}{ll}
P_{A}{}^{C}\bar{P}_{B}{}^{D}S_{CED}{}^{E} & \text{(Ricci-like curvature)}\,,\\
(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD} & \text{(scalar curvature)}\,.
\end{array}
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$$(P^{AC}P^{BD}-\bar{P}^{AC}\bar{P}^{BD})S_{ABCD}\qquad\text{(scalar curvature)}\,.$$

Combining the curvatures, we also have the 'conserved' Einstein-like curvature:

$$\nabla_{A}G^{AB}=0\,,\qquad G^{AB}:=2(P^{AC}\bar{P}^{BD}-\bar{P}^{AC}P^{BD})S_{CED}{}^{E}-\tfrac{1}{2}\mathcal{J}^{AB}(P^{CD}P^{EF}-\bar{P}^{CD}\bar{P}^{EF})S_{CEDF}\,.$$

- *Further completely covariant examples follow from the above generic prescription:*

  - **Completely covariant Yang-Mills field strength** is given by two opposite projections,

$$P_A{}^M \bar{P}_B{}^N \mathcal{F}_{MN} \,,$$

where $\mathcal{F}_{MN}$ is the semi-covariant field strength of a YM potential, $\mathcal{V}_M$,

$$\mathcal{F}_{MN} := \nabla_M \mathcal{V}_N - \nabla_N \mathcal{V}_M - i \left[\mathcal{V}_M, \mathcal{V}_N \right] \,.$$

Unlike the Riemannian case, the Γ connections are not canceled out.

Jeon-Lee-JHP 2011, Choi-JHP 2015

  - **Completely covariant Killing equations** of DFT:

$$\hat{\mathcal{L}}_X \mathcal{H}_{MN} = 0 \quad \Longleftrightarrow \quad (P\nabla)_M (\bar{P}X)_N - (\bar{P}\nabla)_N (PX)_M = 0 \,,$$

$$\hat{\mathcal{L}}_X d = 0 \quad \Longleftrightarrow \quad \nabla_M X^M = 0 \,.$$

JHP-Rey-Rim-Sakatani 2015

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    JHP-Rey-Rim-Sakatani 2015

- ***Apology of semi-covariance:***

  In the ‘Theory of Everything’, one cannot take derivatives arbitraily.

  Taking derivative needs to be disciplined.

- **Now having the semi-covariant differential tool kits at our disposal, we can do various things:**

  - Couple to Yang-Mills, fermions and R-R sector 1011.1324/1109.2035/1206.3478
  - Double Field Theorize the Standard Model 1506.05277
  - Double Field Theorize the Vasiliev's Higher Spin theory 1605.00403
  - Supersymmetrizations
    Half-maximal SDFT 1112.0069
    Maximal SDFT 1210.5078
    Gauged SDFT 1505.01301
    (section condition relaxed)
  - Wald type gravitational Noether currents and global charges 1507.07545
  - Perturbations of $\delta\mathcal{H}_{AB}$, δd 1508.01121
  - Non-Riemannian backgrounds, *e.g.* $\mathcal{H}_{MN} = \mathcal{J}_{MN}$ 1307.8377/1508.01121
  - Extensions to U-duality, 'U-gravity' 1302.1652/1402.5027

Supersymmetric Extension

Utilizing the semi-covariant derivatives introduced above,

after incorporating fermions and R-R sector,

it is possible to construct, to the 'full order' in fermions,

Type II, or $\mathcal{N}=2$, $D=10$ Maximally Supersymmetric Double Field Theory

of which the Lagrangian reads

$$\mathcal{L}_{\mathrm{Max}}=e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}_{\bar{q}}\psi'_{p}\Big]$$

Jeon-Lee-Suh-JHP 1210.5078

c.f. other approaches: Coimbra, Strickland-Constanble, Waldram; Kwak, Hohm, Zwiebach

Symmetries of $\mathcal{N}=2$ $D=10$ SDFT

- **O**$(10,10)$ **T-duality**
- **Gauge symmetries**
  1. **DFT-diffeomorphism (generalized Lie derivative)**
  2. **A *pair* of local Lorentz symmetries, Spin**$(1,9)_L \times$ **Spin**$(9,1)_R$
  3. **local** $\mathcal{N}=2$ **SUSY with** 32 **supercharges.**

- **All the bosonic symmetries are realized manifestly and simultaneously for each term.**
- **For this, it is crucial to have the right field variables:**

$$d\,,\quad V_{Ap}\,,\quad \bar{V}_{A\bar{p}}\,,\quad \mathcal{C}^{\alpha}{}_{\bar{\alpha}}\,,\quad \rho^{\alpha}\,,\quad \rho'^{\bar{\alpha}}\,,\quad \psi^{\alpha}_{\bar{p}}\,,\quad \psi'^{\bar{\alpha}}_{p}$$

which are O$(10,10)$ **covariant genuine DFT-field-variables, and** *a priori* **they are NOT Riemannian, such as metric,** B**-field, R-R** p**-forms.**

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- **Assigning a connection to each (bosonic) gauge symmetry, we introduce 'master' semi-covariant derivative,**

$$\mathcal{D}_A := \partial_A + \Gamma_A + \Phi_A + \bar{\Phi}_A = \nabla_A + \Phi_A + \bar{\Phi}_A\,.$$

- **The spin connections are then determined in terms of the DFT-Christoffel connection by requiring the compatibility with the DFT-vielbeins,**

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Unificaiton of IIA and IIB

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- **The theory is chiral with respect to both Local Lorentz groups:** ρ^{α}, $\psi^{\alpha}_{\bar{p}}$ **vs.** $\rho'^{\bar{\alpha}}$, $\psi'^{\bar{\alpha}}_{p}$

- **Consequently, there is no distinction of IIA and IIB** $\Longrightarrow$ **Unification of IIA and IIB**

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- **Euler-Lagrange equations includes the DFT generalization of the Einstein equation:**

$$S_{p\bar{q}} = -\mathrm{Tr}(\gamma_p \mathcal{F} \bar{\gamma}_{\bar{q}} \bar{\mathcal{F}}) + \text{fermions}\,,$$

where the RR field strength is defined by an O(D, D) covariant nilpotent operator,

$$\mathcal{F} = \mathcal{D}_+ \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} + \gamma^{(11)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A\,, \qquad \mathcal{D}_+^2 = 0\,, \qquad \bar{\mathcal{F}} = \bar{C}_+^{-1} \mathcal{F}^T C_+\,.$$

which covariantizes the H-twisted cohomology, as $\mathcal{D}_+$ reduces to $d_H = d + H\wedge$, upon the diagonal gauge fixing, Spin$(1,9)_L \times$ Spin$(9,1)_R \to$ Spin$(1,9)_D$.

- **Maximal 32 SUSY : covariant Killing Spinor Equations (red color)**

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,, \quad \delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,, \quad \delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon \mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon\rho = -\gamma^p\mathcal{D}_p\varepsilon\,, \qquad \delta_\varepsilon\rho' = -\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\varepsilon'\,,$$

$$\delta_\varepsilon\psi_{\bar{p}} = \mathcal{D}_{\bar{p}}\varepsilon + \mathcal{F}\bar{\gamma}_{\bar{p}}\varepsilon'\,, \qquad \delta_\varepsilon\psi'_p = \mathcal{D}_p\varepsilon' + \bar{\mathcal{F}}\gamma_p\varepsilon\,.$$

** The higher order fermionic terms are suppressed for simplicity above. Nevertheless, the 'full' order supersymmetrization has been completed in* **1210.5078**.

- **Euler-Lagrange equations includes the DFT generalization of the Einstein equation:**

$$S_{p\bar{q}} = -\mathrm{Tr}(\gamma_p \mathcal{F} \bar{\gamma}_{\bar{q}} \bar{\mathcal{F}}) + \text{fermions}\,,$$

where the RR field strength is defined by an O(D, D) covariant nilpotent operator,

$$\mathcal{F} = \mathcal{D}_+ \mathcal{C} := \gamma^A \mathcal{D}_A \mathcal{C} + \gamma^{(11)} \mathcal{D}_A \mathcal{C} \bar{\gamma}^A\,, \qquad \mathcal{D}_+^2 = 0\,, \qquad \bar{\mathcal{F}} = \bar{C}_+^{-1} \mathcal{F}^T C_+\,.$$

which covariantizes the H-twisted cohomology, as $\mathcal{D}_+$ reduces to $d_H = d + H\wedge$, upon the diagonal gauge fixing, Spin$(1,9)_L \times$ Spin$(9,1)_R \rightarrow$ Spin$(1,9)_D$.

- **Maximal 32 SUSY : covariant Killing Spinor Equations (red color)**

$$\delta_\varepsilon d = -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,, \quad \delta_\varepsilon V_{Ap} = i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,, \quad \delta_\varepsilon \bar{V}_{A\bar{p}} = iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,$$

$$\delta_\varepsilon \mathcal{C} = i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,$$

$$\delta_\varepsilon \rho = -\gamma^p \mathcal{D}_p \varepsilon\,, \qquad \delta_\varepsilon \rho' = -\bar{\gamma}^{\bar{p}} \mathcal{D}_{\bar{p}} \varepsilon'\,,$$

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Phenomenological Implication

Standard Model as a Double Field Theory

with Kangsin Choi 1506.05277 PRL

Standard Model as a Double Field Theory : Prediction

- In principle, fermions live on a locally inertial frame, and spin is a gauge symmetry.
- Physically, this local Lorentz symmetry means the arbitrariness in choosing the locally inertial frame at each spacetime point.

- SDFT manifests twofold local Lorentz symmetries: $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$, and as a consequence it unifies type IIA and IIB supergravities.
- Left and right string modes live on two different locally inertial frames. Duff 1986
- **SDFT predicts the fermions in Standard Model are twofold: $\mathbf{Spin}(1, 3)_L \times \mathbf{Spin}(3, 1)_R$.**
  (Even after Scherk-Schwarz compactification, the spin group remains still twofold.)

- Employing the semi-covariant geometry, we can couple the Standard Model to stringy backgrounds in a completely covariant manner: **It is possible to Double Field Theorize the Standard Model, without introducing any extra physical degree.**
- Doing so, one has to decide the spin group for each fermion (*c.f.* Yukawa coupling).
- No experimental evidence of proton decay seems to suggest that **the quarks and the leptons may belong to the different spin classes.**

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Higher Spin Double Field Theory

with Xavier Bekaert 1605.00403

- **The proposed O$(4,4)$ covariant HS-DFT action consists of two parts:**

$$\mathcal{S}_{\mathrm{HS-DFT}} = \int_{\Sigma_4} \mathcal{L}_{\mathrm{HS-DFT}}\,, \qquad \mathcal{L}_{\mathrm{HS-DFT}} = \mathcal{L}_{\mathrm{DFT}} + \mathcal{L}_{\mathrm{HS}}\,,$$

  - **the 'pure' DFT Lagrangian,**

$$\mathcal{L}_{\mathrm{DFT}} = e^{-2d}\Big[(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} - 2\Lambda_{\mathrm{DFT}}\Big]\,,$$

  - **the 'matter' HS Lagrangian, with the Wick normal ordered star product,**

$$\mathcal{L}_{\mathrm{HS}} = g_{\mathrm{HS}}^{-2}\, e^{-2d}\,\mathrm{Tr}\left[P^{AC}\bar{P}^{BD}\mathcal{F}_{AB}\star\mathcal{F}_{CD} + \bar{\Psi}\star\gamma^{(5)}\gamma^{A}\mathcal{D}_{A}\Psi\right]\,.$$

- **The full set of the Euler-Lagrange equations are automatically fulfilled, provided the following 'stronger' equations hold:**

  - **EOMs of the pure DFT,**

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} - 2\Lambda_{\mathrm{DFT}} = 0\,, \qquad P_{A}{}^{C}\bar{P}_{B}{}^{D}S_{CD} = 0\,,$$

  - **BPS-like conditions, as a DFT generalization of the Vasiliev HS equations,**

$$\bar{P}_{A}{}^{B}\mathcal{D}_{B}\Psi = 0\,, \qquad \gamma^{A}\mathcal{D}_{A}\Psi = 0\,,$$
$$P_{A}{}^{C}\bar{P}_{B}{}^{D}\mathcal{F}_{CD} = 0\,, \qquad \left[\Psi^{\alpha},\Psi^{\beta}\right]_{\star}(\mathbf{C}\gamma^{(5)}\gamma^{p})_{\alpha\beta} = 0\,.$$

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Concluding Remarks

- **Equipped with the DFT generalization of the Christoffel connection,**

$$\Gamma_{CAB}=2\big(P\partial_C P\bar{P}\big)_{[AB]}+2\Big(\bar{P}_{[A}{}^{D}\bar{P}_{B]}{}^{E}-P_{[A}{}^{D}P_{B]}{}^{E}\Big)\partial_D P_{EC}-\tfrac{4}{D-1}\Big(\bar{P}_{C[A}\bar{P}_{B]}{}^{D}+P_{C[A}P_{B]}{}^{D}\Big)\Big(\partial_D d+(P\partial^E P\bar{P})_{[ED]}\Big)$$

  we can Double Field Theorize various conventional theories, including SUGRAs, the Standard Model and Higher Spin theory.

- **The 'relaxation' of the section condition is also under control in semi-covariant geometry : the gauged SDFT follows immedinately from the simple replacement,**

$$\left(d,\ V_{Ap},\ \bar{V}_{A\bar{p}}\right)\quad\longrightarrow\quad\left(\dot{d}+\lambda,\ U_A{}^{\dot{A}}\dot{V}_{\dot{A}p},\ U_A{}^{\dot{A}}\dot{\bar{V}}_{\dot{A}\bar{p}}\right)$$

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- *Differential geometry with a projection: Application to double field theory* — 1011.1324 JHEP
- **Stringy differential geometry, beyond Riemann** — **1105.6294 PRD**
- *Incorporation of fermions into double field theory* — 1109.2035 JHEP
- *Ramond-Ramond Cohomology and O(D,D) T-duality* — 1206.3478 JHEP
- *Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity* — 1112.0069 PRD
- **Stringy Unification of IIA and IIB Supergravities under $\mathcal{N}=2$ $D=10$ Supersymmetric Double Field Theory** — **1210.5078 PLB**
- **Supersymmetric gauged Double Field Theory: Systematic derivation by virtue of 'Twist'** — **1505.01301 JHEP**
- *Comments on double field theory and diffeomorphisms* — 1304.5946 JHEP
- **Covariant action for a string in doubled yet gauged spacetime** — **1307.8377 NPB**
- *Double field formulation of Yang-Mills theory* — 1102.0419 PLB
- **Standard Model as a Double Field Theory** — **1506.05277 PRL**
- **O**(D,D) *Covariant Noether Currents and Global Charges in Double Field Theory* — *1507.07545 JHEP*
- *Dynamics of Perturbations in Double Field Theory & Non-Relativistic String Theory* — *1508.01121 JHEP*
- **Higher Spin Double Field Theory: A Proposal** — **1605.00403**
- *U-geometry: SL(5)* $\Rightarrow$ **U-gravity: SL(N)** — 1302.1652 JHEP/**1402.5027 JHEP**
- *M-theory and Type IIB from a Duality Manifest Action* — 1311.5109 JHEP

THE END